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A RIGIDITY THEOREM FOR AUTOMORPHISM GROUPS OF TREES*

BY

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ABSTRACT

A group G is called unsplittable if $\mathrm{Hom}(G, \mathbb{Z}) = 0$ and this group is not a non-trivial amalgam. Let X be a tree with a countable number of edges incident at each vertex and G be its automorphism group. In this paper we prove that the vertex stabilizers are unsplittable groups.

Bass and Lubotzky proved (see [3]) that for certain locally finite trees X, the automorphism group determines the tree X (that is, knowing the automorphism group we can "construct" the tree X). We generalize this Theorem of Bass and Lubotzky, using the above result. In particular we show that the Theorem holds even for trees which are not locally finite.

Moreover, we prove that the permutation group of an infinite countable set is unsplittable and the infinite (or finite) cartesian product of unsplittable groups is an unsplittable group as well.

0. Introduction

In this paper we study the rigidity of automorphism groups of trees.

That is, if X is a tree and $G = \mathrm{Aut}(X)$ is its automorphism group, we study conditions under which the group G determines the tree X. Or, more generally, given two trees X_1, X_2 with $G_1 = \mathrm{Aut}(X_1)$ and $G_2 = \mathrm{Aut}(X_2)$, we study when an isomorphism $G_1 \to G_2$ is induced by an isomorphism $X_1 \to X_2$.

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Such questions, for symmetric spaces, have been studied in E. Cartan's theory (cf. [7]), for homeomorphism groups of topological spaces in [9], and for rooted trees in [10].

Let X be a tree, $G = \mathrm{Aut}(X)$ and let e be an edge starting from the vertex $x = \partial_0(e)$ of X. We set $G_x = \{g \in G : gx = x\}$, $G_e = \{g \in G : ge = e\}$ and $i_G(e) := [G_{\partial_0(e)} : G_e]$. Bass and Lubotzky answered the above questions positively in the case of locally finite trees X with $i_G(e) \geq 3$ for each $e \in EX$. As Bass and Lubotzky observed, the condition of being locally finite is quite restrictive (for example, group amalgams can act on trees which are not locally finite [11]) but necessary for their work (see [3]).

In this paper we work with trees which are not necessarily locally finite. Also, we study the case where we have $i_G(e) \geq 2$ for each $e \in EX$. We restrict ourselves to trees where every vertex has a countable number of edges incident at it. This assumption will hold even if it is not mentioned explicitly.

Our method is the following: Let X be a tree and let $G = \mathrm{Aut}(X)$. We want to "construct" the tree X out of the group G. In particular, we construct a tree Y which is isometric to the tree X, with vertex set $VY = \{G_x : x \in VX\}$ and edge set $EY = \{(G_x, G_y) : x, y \text{ adjacent vertices of } X\}$.

(I) We determine conditions under which the set of the vertex stabilizers $\{G_x : x \in VX\}$ determines the tree X (that is, Y should be a tree isometric to the tree X).

(II) We determine conditions under which the group G determines the set $\{G_x : x \in VX\}$ algebraically. These conditions constitute a set of algebraic properties satisfied only by the groups G_x (of all subgroups of G). Thus, in conjunction with (I) we can construct the tree Y.

The solution in (I) is the condition $i_G(e) \geq 2$ for each $e \in EX$.As for (II), the most important algebraic property which characterizes the set $\{G_x : x \in VX\}$, is that the groups G_x are unsplittable. (A group H is called unsplittable if $\mathrm{Hom}(H, \mathbb{Z}) = 0$ and this group is not a non-trivial amalgam, Section 2).

The steps above are the main steps also used by Bass and Lubotzky in [3]. The main difference between their result and one of the main results of this paper, is that here we prove that the vertex stabilizers of the tree are unsplittable, even in the case where the tree is not locally finite. More specifically:

THEOREM 5.7: *Let X be a tree with a countable number of edges incident at each vertex and let $G = \mathrm{Aut}(X)$. Then for each vertex v of X the group G_v is unsplittable.*

In the Rigidity Theorem of Bass and Lubotzky the condition that the group $G = \mathrm{Aut}(X)$ has no inversions must be added. This is obvious in Corollaries 2.7 and 2.9 in [3], which are used in the proof of the Rigidity Theorem in [3]. This is quite inconvenient because knowing the group G does not mean that its subgroup G^0 can be determined (G^0 as in [3] is the subgroup of G of index ≤ 2 containing no inversions, but containing all stabilizers G_x, $x \in VX$).

This problem is dealt with in this paper by choosing other algebraic properties than those chosen by Bass and Lubotzky which algebraically determine the set $\{G_x : x \in VX\}$ whether the group G has inversions or not.So, we define $A(G)$ to be the set of subgroups $H \leq G$ satisfying:

(i) H is maximal among unsplittable subgroups of G
(ii) H has a countable number of conjugates and
(iii) we have $[H : H \cap K] \neq 2$ for every other such subgroup K.

We prove that $A(G) = \{G_x : x \in VX\}$.

To recover Y, we define $E(G)$ to be the set of elements $(K_1, K_2) \in A(G) \times A(G)$ satisfying:

(i) $K_1 \neq K_2$ and
(ii) the group $K_1 \cap K_2$ is maximal among the supgroups of the form $L_1 \cap L_2$, with $(L_1, L_2) \in A(G) \times A(G)$ and $L_1 \neq L_2$.

Then we prove that $EY = E(G)$. Bearing the above notations in mind, our main result is the following:

RIGIDITY THEOREM 6.9: I) *Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 3$ for each edge e. Then,*
if Y is a tree with $VY = \{G_x : x \in VX\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $X\}$, we have:

(a) $A(G) = VY$
(b) $E(G) = EY$
(c) *The map $\sigma : X \to Y$ with $\sigma(x) = G_x$ is an isomorphism of G-trees (That is, the group G determines the tree X)*
(d) The map $ad : G \to \mathrm{Aut}(G)$ is an isomorphism.

II) *Let* X_1, X_2 *be trees with* $G_1 = Aut(X_1)$, $G_2 = \mathrm{Aut}(X_2)$ *such that* $i_{G_1}(e) \geq 3$ *for each* $e \in EX_1$ *and* $i_{G_2}(w) \geq 3$ *for each* $w \in EX_2$. *Then if* $a : G_1 \to G_2$ *is a group isomorphism, there is a unique tree isomorphism* $\sigma : X_1 \to X_2$ *such that* $a = ad(\sigma)$.

For example, let $X = X_{n,m}$ denote the biregular bipartite tree of degrees $n, m \geq 2$ and $G_{n,m} = \mathrm{Aut}(X)$. A consequence of the Rigidity Theorem if $n, m, n', m' \geq 3$ is that: $G_{n,m} \simeq G_{n',m'} \Leftrightarrow \{n, m\} = \{n', m'\}$.

For $n = m$, $X_n := X_{n,n}$ is the n-regular tree, and the above result is due to Znoiko [13].

The bounds $n, m \geq 3$ have been necessary until now (see introduction in [3]). Indeed, we note that for the trees $X_{n,2}$ and X_n, we have $G_{n,2} \simeq G_{n,n}$, since $X_{n,2}$ is the barycentric subdivision of X_n. However, $X_{n,2}$, X_n do not differ geometrically (that is, they have homeomorphic geometric realizations).

In Section 7, we extend our Rigidity Theorem in the case where $i_G(e) \geq 2$ for every edge e of X. That is, we construct a tree from the group G with geometric realization homeomorphic to that of X. Specifically, the new tree will be obtained from subdivisions of some edges of the tree X.

Before formulating the Topological Rigidity Theorem 7.7 we give a definition: If X is a tree with $G = \mathrm{Aut}(X)$, we define $\overline{X}$ to be the tree obtained from X by subdividing those edges e of X for which there is some $g \in G$ such that $ge = \overline{e}$. In Section 7 we define $\overline{A}(G)$, $\overline{E}(G)$ in a similar way as in (6.9) and we show the following.

Topological Rigidity Theorem 7.7: I) *Let* X *be a tree and let* $G = \mathrm{Aut}(X)$. *We suppose that* X *has a countable number of edges incident at each vertex and* $i_G(e) \geq 2$ *for each edge* e. *Then, if* Y *is a tree with* $VY = \{G_x : x \in V\overline{X}\}$ *and* $EY = \{(G_x, G_y) : x, y$ *adjacent vertices of* $\overline{X}\}$, *we have:*

(a) $\overline{A}(G) = VY$

(b) $\overline{E}(G) = EY$

(c) *The map* $\sigma : \overline{X} \to Y$ *with* $\sigma(x) = G_x$ *is an isomorphism of* G*-trees (That is, the group* G *determines the tree* $\overline{X}$*)*

(d) *The map* $ad : G \to \mathrm{Aut}(G)$ *is an isomorphism.*

II) *Let* X_1, X_2 *be trees with* $G_1 = \mathrm{Aut}(X_1)$, $G_2 = \mathrm{Aut}(X_2)$ *such that* $i_{G_1}(e) \geq 2$ *for each* $e \in EX_1$ *and* $i_{G_2}(w) \geq 2$ *for each* $w \in EX_2$. *Then if* $a : G_1 \to G_2$ *is a group isomorphism, there is a unique tree isomorphism* $\sigma : \overline{X}_1 \to \overline{X}_2$ *such that* $a = ad(\sigma)$.

Referring to the previous example, we have $G_n \simeq G_{n,2}$ and $\overline{X}_n \approx \overline{X}_{n,2}$ ($\overline{X}_{n,2} = X_{n,2}$).

We remark that the restriction $i_G(e) \geq 2$ can not be removed. Indeed, one can easily construct different finite trees, which have trivial automorphism group.

This paper contains also some remarks about unsplittable groups:

We prove in Corollary 5.2 that the infinite (or finite) cartesian product of unsplittable groups is an unsplittable group as well. In Corollary 5.9 we prove that the automorphism group of a rooted tree with a countable number of edges incident at each vertex is an unsplittable group.

Consequently, the permutation group S_k of a set Ω with card $\Omega = k$ for $1 \leq k \leq \infty$ is an unsplittable group. In particular, the wreath product $S_{k_0} \wr S_{k_1} \wr \ldots$, where $k_i \in \mathbb{N}$, $1 \leq k_i \leq \infty$ is an unsplittable group (where the length of the product can be infinite or finite).

Recently, Max Forester [6] also dealt with the Rigidity Theorem of Bass and Lubotzky. He has achieved a generalization of that theorem but from a different point of view.

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1. Tree Automorphisms

1.1. Definitions-Terminology. A set X with a G-action is called a G**-set**. We denote by G_x the set $\{g \in G : gx = x\}$ (i.e. $G_x = \mathrm{stab}_G(x)$) and we set $i_G(x) = [G : G_x] = \mathrm{card}(G \cdot x)$.

A **graph** X consists of a set $V = V(X)$ of vertices and a set $E = E(X)$ of oriented edges. An edge e has end points $\partial_0(e)$, $\partial_1(e) \in V$, and orientation reversal $\overline{e} \in E$, with $\overline{e} \neq e$, $\overline{\overline{e}} = e$ and $\partial_i(\overline{e}) = \partial_{1-i}(e)$, $i = 0, 1$. For $x \in V$ we put $E_0(x) = \{e \in E : \partial_0(e) = x\}$ (the edges starting from x) and $\deg(x) = \mathrm{card}(E_0(x))$.

Let X be a tree. An X**-ray** is a half infinite linear subtree. Two X-rays L, L' are equivalent if $L \cap L'$ is an X-ray. The equivalence classes are called **ends** of X. If ε is one end and $x \in V$, then there is a unique X-ray representing ε starting at x, which we denote by $[x, \varepsilon)$. If $\varepsilon' \neq \varepsilon$ is another end, then the set of vertices $x \in V$ such that $[x, \varepsilon) \cap [x, \varepsilon') = \{x\}$ form the vertices of a bi-infinite linear subtree, denoted by $(\varepsilon, \varepsilon')$.

1.2. Hyperbolic Length of tree automorphisms. Let X be a tree, with automorphism group $G = \mathrm{Aut}(X)$. If $g \in G$, Tits has shown ([12] or [11]) that there are three possible cases:

Case 1 (inversions): There is a (necessarily unique) geometric edge $\{e, \overline{e}\}$ inverted by $g : ge = \overline{e}$. Then for all vertices of X, $d(gx, x) = 2d(x, e) + 1$ is odd. When $g \in G$ is an inversion, we put $l(g) = 0$ and $X_g = \emptyset$. When $g \in G$ is not an inversion, we define $l(g) = min_{x \in V} d(gx, x)$ and $VX_g = \{x \in V : d(gx, x) = l(g)\}$ (VX_g are the vertices of a subtree X_g of X).

Case 2 (elliptic elements): These are non inversions g for which $l(g) = 0$. Then X_g is the tree of fixed points of g. Any $\langle g \rangle$-invariant subtree of X meets X_g.

Case 3 (hyperbolic elements): These are the g's for which $l(g) > 0$. In this case X_g is a bi-infinite linear subtree called the **g-axis** along which g induces a translation of amplitude $l(g)$ toward one of the ends of X_g, which we denote ε_g. Any $\langle g \rangle$-invariant subtree of X contains X_g.

If $g, h \in G$ then $l(ghg^{-1}) = l(h)$, and $X_{ghg^{-1}} = gX_h$. If h is not an inversion, then for all $x \in V$ we have $d(gx, x) = l(g) + 2d(x, X_g)$.

1.3. H-graph. Let H be a group. An H-graph is a graph X with an H-action, given by a homomorphism $p : H \to G = \mathrm{Aut}(X)$.

For $x \in VX$ we have $i_H(x) = [H : H_x] = \mathrm{card}(Hx)$. Moreover H_x acts on $E_0(x)$ and, by abuse of notation, we put $i_H(e) = i_{H_x}(e) = [H_x : H_e] = \mathrm{card}(H_x e)$, where $e \in E_0(x)$.

The H-tree X is said to be without inversions if $\overline{e} \notin He$, for all $e \in EX$. Moreover we have the associated hyperbolic length function $l = l_X : H \to \mathbb{Z}$ defined by $l(g) = l(p(g))$. We also put $X_g = X_{p(g)}$ for $g \in H$. Finally, we set $X^H = \{x \in X : hx = x \ \forall h \in H\}$.

Proposition 1.4 ([12, (3.4)] or [1, (7.5)]): *Let X be an H-tree with $l(H) = \{0\}$ ($l = l_X$). Then exactly one of the following occurs.*

(1) *H fixes some vertex of X.*

(2) *H contains an inversion: $g \in H, e \in EX$ and $ge = \overline{e}$. Then $He = \{e, \overline{e}\}$ and $\{e, \overline{e}\}$ is the unique H-invariant geometric edge of X.*
(3) *There is a unique end ε of X fixed by H. If $x \in VX$ and the ray $[x, \varepsilon)$ has vertex sequence $x_0 = x, x_1, x_2, \ldots$, then $H_{x_n} \leq H_{x_{n+1}}$, with strict inclusion infinitely often, and $H = \bigcup_{n\geq 0} H_{x_n}$.*

Definition 1.5: Let X be a tree and ε be an end of X. We put

$$H_\varepsilon = \{g \in H : g\varepsilon = \varepsilon\}$$

the stabilizer of the end ε.

2. Unsplittable groups

Proposition 2.1 ([2, 3.9]): *Let H be a group. The following conditions are equivalent:*

(1) *For any H-tree X, $l_X(H) = \{0\}$*
(2) *For any H-tree X without inversions, each element of H fixes some vertex of X.*
(3) (a) *$Hom(H, \mathbb{Z}) = 0$ and*
 (b) *H is not a non-trivial amalgam, that is, if $H \cong A *_C B$ then $C = A$ or $C = B$.*

Definition 2.2: A group satisfying the conditions of Propositions (2.1) is said to be **unsplittable**.

Proposition 2.3 ([11, Prop. 27, Chap. 6]): *Let G be a finitely generated nilpotent group acting without inversions on a tree X. Then the next two cases are mutually exclusive and exhaust all possibilities:*

(a) *G has a fixed point.*
(b) *There is an X-ray stable under G, on which G acts by translations by means of a non-trivial homomorphism $G \to \mathbb{Z}$.*

Corollary 2.4: *A_i are unsplittable groups and*

$$G = \times_i A_i = \{(g_1, g_2, \ldots, g_n, 1, 1, \ldots) : g_i \in A_i, n \in \mathbb{N}\},$$

then G is unsplittable.

Proof. Let G act on a tree X without inversions. Then A_i will act on X as well without inversions. Let $g = g_1 g_2 \cdots g_n = (g_1, g_2, \ldots g_n, 1, 1, \ldots) \in G$, where the element $g_i \in A_i$ is identified with the element $g_i = (1, \ldots, 1, g_i, 1, \ldots) \in G$ (with g_i in the i place). Since $A_1, A_2, \ldots, A_n$ are unsplittable then $g_1, \ldots, g_n$ are elliptic. Then, applying Proposition 2.3 to the group $\langle g_1 \rangle \times \cdots \times \langle g_n \rangle$ we have that there is $v \in VX$ such that $g_i v = v$, $i = 1, 2, \ldots, n$ and so $gv = v$, that is, G is an unsplittable group. ∎

3. Strict action

Definition 3.1: Let X be an H-set. It is said to be a strict H-set if any of the following equivalent conditions holds:

(a) $X^{H_x} = \{x\}$ for each $x \in X$, where X^{H_x} is the set of fixed points by H_x.

(b) If $x, y \in X$ such that $H_x \leq H_y$ then $x = y$.

Definition 3.2: An H-graph X is called strict if:

(a) VX is a strict H-set, and

(b) for all $x \in VX$, $E_0(x)$ is a strict H_x-set.

LEMMA 3.3 (see [3]): *Let X be an H-tree.*

(a) *The following conditions are equivalent:*

  (1) *VX is a strict H-set.*

  (2) *$i_H(e) > 1$, for all $e \in EX$.*

  (3) *For all $x \in VX$, H_x fixes no edges of X.*

(b) *If $x \in VX$ is not a terminal vertex, that is, if $\deg(x) \neq 1$ and if $E_0(x)$ is a strict H_x-set, then $i_H(e) \geq 3$ for all $e \in E_0(x)$.*

(c) *If X has no terminal vertices then the condition "VX is a strict H-set" is implied by the condition "for all $x \in VX$, $E_0(x)$ is a strict H_x-set".*

The following Proposition is a special case of Proposition 3.7[3].

PROPOSITION 3.4: *If X is a tree and $G = \operatorname{Aut} X$ with $i_G(e) \geq 3$, then X is a strict G-tree.*

4. Geometrical Tree determination

In this section we show that the set $\{G_x : x \in VX\}$ determines the tree X under the condition $i_G(e) \geq 2$.

4.1. Terminology. Let X be a tree and $G = \mathrm{Aut}(X)$ with $i_G(e) \geq 2$ for all $e \in EX$. We define a G-graph Y with set of vertices $VY = \{G_x : x \in VX\}$ and set of edges $EY = \{(G_x, G_y) : x, y$ adjacent vertices of X $\}$.

Proposition 4.2: *Using the above notation we define a map $f : X \to Y$, with $f(x) = G_x$ and $f(e) = (G_{\partial_0(e)}, G_{\partial_1(e)})$ for all $x \in VX, e \in EX$. Then, f is a tree isomorphism, where Y is regarded as a G-graph via conjugation.*

Proof. Since VX is a strict G set (from Lemma 3.3), one can easily prove that Y is a G-tree and f is a G-graph morphism. From the strictness of VX as a G-set, it follows that f is one to one on the vertices. Now, if $f(e) = f(w)$ with $e, w \in EX$, it follows that $(G_{\partial_0(e)}, G_{\partial_1(e)}) = (G_{\partial_0(w)}, G_{\partial_1(w)})$. Therefore, $\partial_0(e) = \partial_0(w)$ and $\partial_1(e) = \partial_1(w)$. Since X is a tree we have $e = w$, that is, f is one to one on the edges also. It is clear that f is onto on the edges and vertices alike. Finally, the action defined above is the one which is induced on Y by f, so f is a G-tree isomorphism. ∎

5. Stabilizers of tree vertices

In this section we study the vertex stabilizers of trees with a countable number of edges incident at each vertex and we prove that these groups are unsplittable.

Lemma 5.1: *Let $(G_i)_{i \in \mathbb{N}}$ be a family of groups and let*

$$G = \prod_{i \in \mathbb{N}} G_i = \{(g_1, g_2, \ldots)/g_i \in G_i, i = 1, 2, \ldots\}.$$

Suppose that G acts on a tree T without inversions. If there are elements $g_i \in G_i$ which are elliptic for each $i \in \mathbb{N}$, then the element $g = (g_1, g_2, g_3, \ldots)$ is elliptic too (where the element $g_i \in G_i$ is identified with the element $g_i = (1, \ldots, 1, g_i, 1, \ldots) \in G$, with g_i in the i place).

Proof. We remark that if p is an odd prime then $2^{n_p} \equiv 1 \, mod\, p$ for some $n_p \in \mathbb{N}$ (i.e. $n_p = p - 1$). Clearly, if $k \in \mathbb{N}$ then $2^{kn_p} \equiv 1 \, mod\, p$. If $\{p_1, p_2, \ldots\}$ are the odd primes we consider the sequence $n_i = n_{p_1} n_{p_2} \cdots n_{p_i}$, $i \in \mathbb{N}$. We have then $2^{n_i} \equiv 1 mod\, p_k$ for all $k \leq i$.

We consider now the element $h = (g_1^{2^{n_1}}, \ldots, g_i^{2^{n_i}}, \ldots)$ and we prove that it is elliptic: We suppose the opposite, that is, h is hyperbolic. If we identify the element $g_i^{2^{n_i}} \in G_i$ with the element $(1, \ldots, 1, g_i^{2^{n_i}}, 1, \ldots)$ (with $g_i^{2^{n_i}}$ in the i place), then $h = g_1^{2^{n_1}} g_2^{2^{n_2}} \ldots g_{k-1}^{2^{n_{k-1}}} h_k$ where $h_k = (1, 1, .., 1, g_k^{2^{n_k}}, g_{k+1}^{2^{n_{k+1}}}, \ldots)$

for each $k \in \mathbb{N}$. So, from Proposition 2.3, h_k is a hyperbolic element with the same translation length as h (so we can "forget" the first coordinates). But h_k is a 2^{n_k}-th power so its translation length is a multiple of 2^{n_k}. So 2^{n_k} divides the translation length of h for all k, which is impossible. Now we suppose that g is hyperbolic, so from Proposition 2.3 hg^{-1} is hyperbolic too. Since $hg^{-1} = (g_1^{2^{n_1}-1}, \ldots, g_i^{2^{n_i}-1}, \ldots)$ and p_i divides $2^{n_i} - 1$, we have as before that p_i divides the translation length of hg^{-1} for all i, which is impossible. So g is elliptic. ∎

COROLLARY 5.2: *If the groups G_i are unsplittable for each $i \in \mathbb{N}$ then the group $G = \prod_{i\in\mathbb{N}} G_i = \{(g_1, g_2, \ldots)/g_i \in G_i, i = 1, 2, \ldots\}$ is unsplittable too.*

Proof. We suppose that G acts on a tree T without inversions. Let $g = (g_1, g_2, \ldots) \in G$. We prove that the element g is elliptic. Since G_i is unsplittable, g_i is elliptic for each $i \in \mathbb{N}$. So from Lemma 5.1 the element g is elliptic too. ∎

LEMMA 5.3: *Let X be a tree, $G = \mathrm{Aut}(X)$ and v a vertex of X. We suppose that we have pairs (X_i, v_i), where X_i is a subtree of X and v_i a vertex of X_i (root of X_i), for all $i \in \mathbb{N}$ such that $VX_i \cap VX_j = \{v_i\} \cap \{v_j\}$ for all $i, j \in \mathbb{N}$. If G_v acts without inversions on a tree T, then there is no sequence of elements $g_i \in G_v \cap G_{v_i}$ for all $i \in \mathbb{N}$, where $g_i(x) = x$ for all $x \notin VX_i$ (that is, every g_i acts non-trivially only on X_i) and g_i is hyperbolic for each $i \in \mathbb{N}$.*

Proof. Let us suppose the opposite. Let $l(g_k) := l_k > 0$ (see (1.2)). Since the g_i's commute, from Lemma 2.3 they have the same axis L. Replacing some g_i's by ${g_i}^{-1}$ if necessary we can arrange that all g_i's act on L by translation in the same direction. We put $n_{k+1} = k(l_1 + l_2 + \cdots + l_k + 1)n_k$ for all $k \in \mathbb{N}$ (i.e. $n_1 = 2$) and

$$h(x) = \begin{cases} g_k^{n_k}(x), & \text{if } \exists\, k \in \mathbb{N} : \ x \in VX_k \\ x, & \text{if } x \in VX - \bigcup_{i\in\mathbb{N}} VX_i \end{cases} \tag{1}$$

$$h_k(x) = \begin{cases} g_r^{n_r}(x), & \text{if } \exists\, r \geq k : \ x \in VX_r \\ x, & \text{if } x \in VX - \bigcup_{i\geq k} VX_i. \end{cases} \tag{2}$$

Let $l(h) = k_0 \geq 0$. It is easy to see that

$$h = g_1^{n_1} g_2^{n_2} \cdots g_{k_0}^{n_{k_0}} h_{k_0+1} \tag{3}$$

and that the elements $g_i^{n_i}$ and h_{k_0+1} commute for all $i \leq k_0$. Since all $g_i, i = 1, 2, \cdots, k_0$, act on L by translation in the same direction, we have that $l(g_1^{n_1} g_2^{n_2} \cdots g_{k_0}^{n_{k_0}}) = n_1 l_1 + \cdot + n_{k_0} l_{k_0} > k_0$. From (3) and Lemma 2.3 we have that h_{k_0+1} is hyperbolic and translates L in the opposite direction from that of $g_i^{n_i}$. But h_{k_0+1} is a power of n_{k_0+1} (since n_i/n_{i+1}) and so $l(h_{k_0+1})$ is a multiple of n_{k_0+1}. Moreover, $n_{k_0+1} = k_0(l_1 + \cdots + l_{k_0} + 1)n_{k_0} > l(g_1^{n_1} \cdots g_{k_0}^{n_{k_0}}) + l(h)$ so from (3) we have that $l(h_{k_0+1}) - l(g_1^{n_1} \cdots g_{k_0}^{n_{k_0}}) = k_0$ which is a contradiction. ∎

5.4. Terminology. Let X be a tree and $v, w \in VX$. We define $X_{[v,w]}$ to be the maximal subtree of X which contains the vertex w and does not contain any other vertex of the path $[v, w]$. If $A \subseteq E_0(v)$, we define $X_{(v,A)}$ to be the subtree of X which contains the vertex v and every vertex w such that the first edge of the path $[v, w]$ starting from v belongs to A. We denote by $O_1^v, O_2^v, \ldots$ the orbits of $E_0(v)$ under the action of G_v. If $g \in G_v$ we define $g_i \in G_v$ by

$$g_i(x) = \begin{cases} g(x), & \text{if } x \in VX_{(v,O_i^v)} \\ x, & \text{if } x \notin VX_{(v,O_i^v)} \end{cases} \tag{4}$$

Finally, we put $G_{v,i} = \mathrm{Aut}(X_{(v,O_i^v)})_v$.

Definition 5.5: Let l be an infinite path of a tree X (a half line) starting from vertex w and with vertex sequence $w_0 = w, w_1, w_2, \ldots$. If we denote by e_i the edge $[w_i, w_{i+1}]$ then we say that the edge e_i has the property P if there is some $g \in \mathrm{Aut}(X_{[w,w_i]})_{(w_i)}$ such that $ge_i \neq e_i$ ($ge_i \notin l$).

Remark 5.6: Let e_i be an edge of the path l (as above). If on the tree $X_{[w,w_i]}$ there are card(I_i) orbits of edges ($I_i \subseteq \mathbb{N}$) starting from the vertex w_i, under the action of group $\mathrm{Aut}(X_{[w,w_i]})_{(w_i)}$ then, we denote these orbits by $O_j^{[w,w_i]}, j \in I_i$. In the case where the edge e_i does not have the property P, there is a unique $j_0 \in I_i$ such that $O_{j_0}^{[w,w_i]} = \{e_i\}$.

Also, we denote by $X_{[w_0,w_i]}^j$ the tree $X_{(w_i, O_j^{[w_0,w_i]})}$ and by $G_{w_0,w_i,j}$ the vertex group $(\mathrm{Aut}\, X_{[w_0,w_i]}^j)_{(w_i)}$.

Theorem 5.7: *Let X be a tree with a countable number of edges incident at each vertex and let $G = \mathrm{Aut}(X)$. Then for each vertex v of X the group G_v is unsplittable.*

Proof. Let the group G_v act on a tree T without inversions and $g \in G_v$ be a hyperbolic element.

CLAIM 1: *There is a hyperbolic element of G_v which acts non trivially only on some $X_{(v,O_i^v)}$.*

Proof of Claim 1. It is obvious that $G_v = \prod_{i\in\mathbb{N}} G_{v,i}$. If g_i is the restriction of g on the tree $X_{(v,O_i^v)}$ as in (5.4)(4), then $g = (g_1, g_2, \ldots)$. From Lemma (5.1) we may suppose that some of g_i's are hyperbolic. Let g_1 be hyperbolic (acting only on $X_{(v,O_1^v)}$). ∎

We prove that $\text{card}(O_1^v) \neq \aleph_0$. Suppose that $\text{card}(O_1^v) = \aleph_0$.

CLAIM 2: *There is a hyperbolic element of G_v which acts non-trivially only on some subtree $X_{(v,A)}$ where $A \subseteq O_1^v$ and the set $O_1^v - A$ is infinite.*

Proof of Claim 2. The restriction of g_1 on O_1^v is a permutation on the set O_1^v that is, an element of the group $S_\infty = symm(O_1^v)$. If the restriction of g_1 on O_1^v is a finite permutation, then there is $k \in \mathbb{N} : g_1^k = 1$ on O_1^v. If $v_0, v_1, v_2, \ldots$ are the adjacent vertices of v, such that $[v, v_i] \in O_1^v$, then $g_1^k \in \prod_{i\in\mathbb{N}} \text{Aut}(X_{[v,v_i]})_{(v_i)}$. From (5.1) there is a hyperbolic element $h \in \text{Aut}(X_{[v,v_{i_0}]})_{(v_{i_0})}$ for some $i_0 \in \mathbb{N}$. Therefore, $A = \{[v, v_{i_0}]\}$ and h is the required hyperbolic element.
If the restriction of g_1 on O_1^v is permutation with infinite support, then either the restriction of g_1 or the restriction of g_1^2 on O_1^v will be a product of two disjoint permutations $a, b \in \text{symm}(O_1^v)$ with infinite support. Indeed, the restriction of g_1 on O_1^v is written as a product of disjoint cycles on $\text{symm}(O_1^v)$. We distinguish two cases for these cycles.

CASE 1: Among these cycles there is at least one cycle with infinite support.

In this case, since the square of a cycle with infinite support is a product of disjoint cycles with infinite support, the restriction of g_1^2 on O_1^v is written as a product of two disjoint permutations $a, b \in \text{symm}(O_1^v)$ with infinite support.

CASE 2: Among these cycles there are no cycles with infinite support.

In this case there are $\aleph_0$ such cycles. Then each one of the required permutations a, b can be written as an infinite product of such cycles. Therefore, in each case, the restriction of g_1 or the restriction of g_1^2 on O_1^v will be a product of two disjoint permutations $a, b \in \text{symm}(O_1^v)$ with infinite support. We put

$A_1 = \text{support}(a)$ (the elements of O_1^v which are moved by a), $A_2 = \text{support}(b)$. Then the element g_1 or g_1^2 can be written as a product of two elements $h, f \in G_v$ where h acts non-trivially only on $X_{(v,A_1)}$ and f acts non trivially on $X_{(v,A_2)}$ as well. From Lemma 2.3, at least one of f, h is hyperbolic and hence the proof of the claim above is complete. ∎

Let h be the hyperbolic element obtained above. Since $\text{card}(O_1^v - A) = \aleph_0$, we can write the set $O_1^v - A$ as disjoint union of sets $\Omega_1, \Omega_2, \ldots$, where these sets have the same cardinal number as the set A. Since O_1^v consists of one orbit of edges, we have that for any $i \in \mathbb{N}$ there is some $t_i \in G_v$ such that h^{t_i} ($h^{t_i} = t_i^{-1} h t_i$) acts non-trivially only on $X_{(v,\Omega_i)}$. Now, if we apply Lemma 5.3 on pairs $(X_{(v,\Omega_i)}, v)$ it follows that, there is $i \in \mathbb{N}$ such that h^{t_i} is an elliptic element. But this is a contradiction, since h is hyperbolic.

Let us suppose now that $\text{card}(O_1^v) = k < \infty$ and let $v_1, \ldots, v_k$ be the end points of the edges of the set O_1^v (which start from v). Obviously there is some $r \leq k$ such that $g_1^r = 1$ on O_1^v and so $g_1^r = h_1 h_2 \cdots h_k$, where

$$h_i(x) = \begin{cases} g_1^r(x), & if x \in X_{[v,v_i]} \\ x, & \text{otherwise} \end{cases} \tag{5}$$

(h_i is the restriction of g_1^r on $X_{[v,v_i]}$). The elements h_i commute among each other and therefore from Lemma (2.3) at least one of them will be hyperbolic. Let h_1 be hyperbolic. Then, as in Claim 1, for the tree $X_{[v,v_1]}$, there is a hyperbolic element of $\text{Aut}(X_{[v,v_1]})_{(v_1)} \leq G_v$ which acts on a tree of the form $Y_{(v_1,R)}$, where $Y = X_{[v,v_1]}$ and R is an orbit of edges of the set $E_0(v_1)$ on Y, under the action of group $\text{Aut}(X_{[v,v_1]})_{(v_1)}$. As above, R must be finite. Working in the same way, we create a path l with vertices $w_0 = v, w_1 = v_1, w_2, \ldots$ and a sequence of hyperbolic elements f_i of G_v where every f_i acts non trivially only on the tree $X_{[w_0,w_i]}$ ($i = 1, 2, \ldots$). We denote by e_i the edge $[w_i, w_{i+1}]$ and we distinguish two cases:

Case 1: There is a subsequence $(e_{n_k})_{k\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$, where all of its terms have the property P. Therefore, for all $k \in \mathbb{N}$ there is an edge $r_{n_k} \in E_0(w_{n_k}) - l$, such that e_{n_k}, r_{n_k} belong to the same orbit under the action of group $\text{Aut}(X_{[w_0,w_{n_k}]})_{(w_{n_k})}$.

In this case, there is some $t_k \in \text{Aut}(X_{[w_0,w_{n_k}]})_{(w_{n_k})}$ such that, the hyperbolic element $f_{n_k+1}^{t_k}$ acts non-trivially only on the tree $X_{[w_0,\partial_1(r_{n_k})]}$. If we apply

Lemma 5.3, on the pairs $(X_{[w_0,\partial_1(r_{n_k})]}, \partial_1(r_{n_k}))$ at least one of the elements $f_{n_k+1}^{t_k}$ must not be hyperbolic. But this is a contradiction.

Case 2: There is no subsequence of $(e_n)_{n\in\mathbb{N}}$, where all of its terms have the property P. In this case, there is some $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have $ge_i = e_i$ for all $g \in \mathrm{Aut}(X_{[w_0,w_i]})_{(w_i)}$. That is, on the path with vertex sequence $w_{i_0}, w_{i_0+1}, w_{i_0+2}, \ldots$, the edges e_i for $i \geq i_0$ don't have the property P. So $f_{i_0+1} \in \prod_{j\in I_{i_0+1}-\{j_0\}} G_{w_{i_0},w_{i_0+1},j}$ (where j_0 as in 5.6). Since f_{i_0+1} is hyperbolic, from (5.1) there is $j_1 \in I_{i_0+1} - \{j_0\}$ and a hyperbolic element $f_{(i_0+1,j_1)} \in G_{w_{i_0},w_{i_0+1},j_1}$ with $e_{i_0+1} \notin O_{j_1}^{[w_0,w_{i_0+1}]}$.

Working in the same way, we construct a sequence of hyperbolic elements $(f_{(i_0+k,j_k)})_{k\geq 1}$ where every $f_{(i_0+k,j_k)}$ acts non-trivially only on the tree $X^{j_k}_{[w_{i_0},w_{i_0+k}]}$. We obtain again a contradiction by (5.3). ∎

Before the main application of Theorem 5.7, that is, Theorem 6.9, we mention a simple application of this Theorem on rooted trees.

5.8. Rooted trees. A rooted tree (T, v_0) is a tree with a fixed vertex v_0 named the root of the tree. An automorphism f of the rooted tree (T, v_0) is an automorphism of the tree T such that $f(v_0) = v_0$. We denote the automorphism group of the rooted tree (T, v_0), $\mathrm{Aut}(T, v_0)$. We have that, $\mathrm{Aut}(T, v_0) = \mathrm{Aut}(T)_{(v_0)}$. We put $V_n = \{v \in V(T) : d(v_0, v) = n\}$. If $v \in V_n$ we call "children" of v the vertices of V_{n+1} which are adjacent to v. We denote by $c(v)$ the number of children of the vertex v. A rooted tree (T, v_0) of the form $(k_0, k_1, k_2, \ldots)$ (where $k_i \in \mathbb{N}$, $0 \leq k_i \leq \infty$) is a rooted tree where, for each $n \in \mathbb{N}$, we have $c(v) = k_n$ for all $v \in V_n$.

Corollary 5.9: (1) *If (T, v_0) is a rooted tree with a countable number of children incident at each vertex, then $\mathrm{Aut}(T, v_0)$ is an unsplittable group.*

(2) *The group S_∞ is unsplittable*

(3) *The wreath product $S_{k_0} \wr S_{k_1} \wr \ldots$, where $k_i \in \mathbb{N}$, $0 \leq k_i \leq \infty$ is an unsplittable group (where the length of the product can be infinite or finite).*

Proof. (1) It follows from Theorem 5.7, since $\mathrm{Aut}(T, v_0) = \mathrm{Aut}(T)_{(v_0)}$.

(2) It follows from (1) that if we take (T, v_0) to be the rooted tree of the form $(k_0, k_1, \ldots)$, where $k_0 = \infty$ and $k_i = 0\ \forall i \geq 1$. Then $S_\infty = \mathrm{Aut}(T, v_0)$.
(3) Similarly, $S_{k_0} \wr S_{k_1} \wr \ldots = \mathrm{Aut}(T, v_0)$, where (T, v_0) is of the form $(k_0, k_1, \ldots)$ ∎

6. Rigidity Theorem

In this section we show that the group $G = \mathrm{Aut}(X)$ determines the set $\{G_x : x \in VX\}$ and therefore the tree X as well, where X is a tree such that $i_G(e) \geq 3$ for each edge e.

6.1. Terminology. Let X be a tree and $x_0 \in VX$. For the rooted tree (X, x_0) we number the children of each vertex and thus we identify the vertices with finite sequences as follows : if $x \in V_r$ (as in Subsection 5.8) and $i_1, \ldots, i_r$ are the branches we follow going from x_0 to x, then we put $x = (i_1, \ldots, i_r)$.

Proposition 6.2: *If X_3 is the homogeneous tree of degree 3 at each vertex, and ε_0 is an end, then the set $\{G_{g\varepsilon_0} : g \in G\} = \{G_\varepsilon : \varepsilon$ an end of $X_3\}$ is uncountable.*

Proof. Since the tree is homogeneous, it is obvious that $\{G_{g\varepsilon_0} : g \in G\} = \{G_\varepsilon : \varepsilon$ an end of $X_3\}$. If $x_0 \in VX_3$, and we number the branches as in (6.1) then the X-rays $[x_0, \varepsilon)$ correspond to the sequences of the form $(i_1, i_2, \ldots)$ with $i_k \in \{1, 2\}$, for $k \geq 2$. Therefore, there are uncountable many such rays.

The above X-rays represent distinguished ends, therefore X_3 has uncountably many ends. For two such ends ε_1, ε_2 it is obvious that we have $G_{\varepsilon_1} \neq G_{\varepsilon_2}$. We remark that G_{ε_1}, G_{ε_2} are conjugate. Thus, the set $\{G_\varepsilon : \varepsilon$ an end of $X_3\}$ is uncountable. ∎

Corollary 6.3: *Let X be a tree with $i_G(e) \geq 3$ for each $e \in EX$, and let ε be an end. Then the group G_ε has an uncountable number of conjugate subgroups.*

Corollary 6.4: *Let X be a tree with a countable number of edges incident at each vertex and let $G = \mathrm{Aut}(X)$ with $i_G(e) \geq 3$ for each edge e. Then, if $x \in VX$, the group G_x has a countable number of conjugate subgroups.*

Proof. We define a map $f : \{gG_xg^{-1} : g \in G\} = \{G_{gx} : g \in G\} \to VX$ with $f(G_{gx}) = gx$. From Proposition 3.4 the tree X is a strict G-tree. Therefore, f

is properly defined and it is apparently one to one, so the set $\{gG_xg^{-1} : g \in G\}$ is countable. ∎

LEMMA 6.5: *Let X be a tree with a countable number of edges incident at each vertex and let $G = \mathrm{Aut}(X)$ with $i_G(e) \geq 3$ for each edge e. If K is an unsplittable subgroup of G containing inversions, then K fixes some geometrical edge $\{e, \bar{e}\}$. If K is also maximal unsplittable, then $K = G_{\{e,\bar{e}\}}$ i.e. K is the stabilizer of a geometrical edge.*

Proof. The subgroup K acts on the barycentric subdivision X' of X. From Proposition 1.4 and the fact that K contains inversions of the tree X, we have that K fixes a vertex $v \in VX' - VX$. Therefore, $K \subseteq G_{\{e,\bar{e}\}}$, for the corresponding geometrical edge. From Theorem 5.7 we have that G'_v is unsplittable ($G' = \mathrm{Aut}(X')$) and thus the same holds for $G_{\{e,\bar{e}\}}$. Therefore, if K is maximal unspilttable then $K = G_{\{e,\bar{e}\}}$. ∎

Definition 6.6: Let G be a group. Then we define

(a) $A(G)$ to be the set of subgroups $H \leq G$ satisfying:
  (i) H is maximal among unplittable subgroups of G
  (ii) H has a countable number of conjugates and
  (iii) we have $[H : H \cap K] \neq 2$ for every other such subgroup.
(b) $E(G)$ to be the set of elements $(K_1, K_2) \in A(G) \times A(G)$ satisfying:
  (i) $K_1 \neq K_2$ and
  (ii) the group $K_1 \cap K_2$ is maximal among the subgroups of the form $L_1 \cap L_2$, with $(L_1, L_2) \in A(G) \times A(G)$ and $L_1 \neq L_2$.

PROPOSITION 6.7 (Separation of vertices): *Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 3$ for each edge e. Then, if Y is a tree with $VY = \{G_x : x \in VX\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $X\}$, we have $A(G) = VY$.*

Proof. Let $x \in VX$. Then, from Theorem 5.7 G_x is unsplittable. The group G_x is also maximal with this property. Indeed, if we suppose that $G_x \leq K$, with K being unsplittable, we show that $G_x = K$. Since K is unsplittable from Proposition 1.4 then one of the following cases holds:

The first case is that K fixes some vertex y, so $G_x \leq K \leq G_y$ and since VX is a strict G-set ($i_G(e) \geq 3$), it follows that $x = y$, therefore $G_x = K$.

The second case is that K fixes a unique end ε of X, and so $K = \bigcup_y K_y$ for y along $[x, \varepsilon)$ approaching ε. For such a y, we have $G_x \leq K_x \leq K_y \leq G_y$ and from strictness we have $x = y$. Thus, $K_y = G_x$ for all these y, and so $K = G_x$.

The third case is that K contains an inversion and then $K \subseteq G_{\{e,\bar{e}\}}$ for some geometrical edge (from Lemma 6.5). But then $G_x \leq G_{\{e,\bar{e}\}}$, which does not hold in view of strictness. Also, for each $x, y \in VX$, we have that $[G_x : G_x \cap G_y] \geq 3$ if $x \neq y$ and finally from Corollary 6.4, we have that $VY \subset A(G)$.

Now, if $H \in A(G)$ from Proposition 1.4 we have the following cases:

(a) $H \leq G_x$, and since G_x is unsplittable, we have that $H = G_x$

(b) $H \leq G_\varepsilon$ (for a unique end) and then $H^g \leq G_{g\varepsilon}$ $\forall g \in G$. But the set $\{G_{g\varepsilon} : g \in G\}$ is uncountable and so there is $g \in G\text{-}G_\varepsilon$, with $H^g = H$. So $H \leq G_{g\varepsilon} \cap G_\varepsilon$, with $g\varepsilon \neq \varepsilon$. That is, H fixes two different ends which does not hold from Proposition 1.4.

(c) If H has inversions, then from Lemma 6.5, we have that $H = G_{\{e,\bar{e}\}}$. But then for $x = \partial_0(e), K = G_x$ we have $[H : H \cap K] = 2$, which does not hold. So $VY = A(G)$. ∎

Proposition 6.8 (Separation of edges): *Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 3$ for each edge e. Then, if Y is a tree with $VY = \{G_x : x \in VX\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $X\}$, we have $E(G) = EY$.*

Proof. Let $(G_x, G_y) \in EY$, that is, x, y are the extremities of an edge e (from strictness we have $G_x \neq G_y$). If $x' \neq y'$ are vertices and $G_x \cap G_y \leq G_{x'} \cap G_{y'}$, then $G_e = G_x \cap G_y =: G_{x,y} \leq G_{x'}$. We will show that x' is an extremity of e.

If x' is not an extremity of e, then $y \in [x, x']$ or $x \in [y, x']$. Supposing the second assumption holds, there is an edge $w \neq e$ in $[x, x']$, with x being the extremity. That is, $w, e \in E_0(x)$. Since $G_e \leq G_{x'}$, we have that $G_e \leq G_w$ and from the strictness of $E_0(x)$ (see Proposition 3.4), we have $e = w$, which does not hold. Therefore, x' is an extremity of e and so is y', that is, $\{x, y\} = \{x', y'\}$. Thus, $EY \subseteq E(G)$.

Now, if $(G_x, G_y) \in E(G)$ (also $x \neq y$), we will show that x, y are adjacent. If we suppose the opposite, then for each edge $e \in [x, y]$, we have that $G_x \cap G_y \leq G_{x'} \cap G_{y'}$, where x' and y' are the extremities of e. But we supposed that $(G_x, G_y) \in E(G)$, and so $G_{x,y} = G_{x',y'}$. Therefore, $G_e \leq G_x$ and $G_e \leq G_y$, whence $\{x, y\} = \{x', y'\}$, that is, $e = [x, y]$. Thus x, y are adjacent and therefore $E(G) \subseteq EY$. ∎

Rigidity Theorem 6.9: I) *Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 3$ for each edge e. Then, if Y is a tree with $VY = \{G_x : x \in VX\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $X\}$, we have:*

(a) $A(G) = VY$.

(b) $E(G) = EY$.

(c) *The map $\sigma : X \to Y$ with $\sigma(x) = G_x$ is an isomorphism of G-trees. (That is, the group G determines the tree X.)*

(d) *The map $ad : G \to \mathrm{Aut}(G)$ is an isomorphism.*

II) *Let X_1, X_2 be trees with $G_1 = \mathrm{Aut}(X_1)$, $G_2 = \mathrm{Aut}(X_2)$ such that $i_{G_1}(e) \geq 3$ for each $e \in EX_1$ and $i_{G_2}(w) \geq 3$ for each $w \in EX_2$. Then if $a : G_1 \to G_2$ is a group isomorphism, there is a unique tree isomorphism $\sigma : X_1 \to X_2$ such that $a = ad(\sigma)$.*

Proof. I) (a),(b) and (c) follow immediately from Propsition 4.2,Proposition 6.7 and Proposition 6.8. (d) It suffices to show that ad is one to one and onto.

As far as onto is concerned: If $a \in \mathrm{Aut}\, G$ it is obvious that $K \in A(G)$ if and only if $a(K) \in A(G)$ and $(K, L) \in E(G)$ if and only if $(a(K), a(L)) \in E(G)$. Therefore, for each $x \in VX$ there is unique $x' \in VX$ such that $a(G_x) = G_{x'}$ (due to strictness), and we put $a'(x) = x'$. That is, the map $a' : VX \to VX$ is properly defined and preserves the adjacency of the vertices. Therefore the map a'extends to an automorphism of tree X.

We observe that we have $a(G_{gx}) = G_{a(g)a'(x)}$, so $a'(gx) = a(g)a'(x)$ for each $g \in G$, $x \in VX$, (1).

Now we can show that $ad(a') = a$: If $g \in G$, $x \in VX$, then from (1) we have that $(ad(a')(g))(a'(x)) = a'(gx) = a(g)a'(x)$. Therefore, $ad(a')(g) = a(g)$, and so $ad(a') = a$.

The map ad is one to one: That is, if $a, b \in \mathrm{Aut}(G)$, with $a = b$, we show that $a' = b'$. Indeed, if $x \in VX$, then $a(G_x) = b(G_x)$, therefore $G_{a'(x)} = G_{b'(x)}$. Thus, $a'(x) = b'(x)$, and so $a' = b'$.

II) If $a : G_1 \to G_2$ is a group isomorphism, it is obvious that $K \in A(G_1)$ if and only if $a(K) \in A(G_2)$ and $(K, L) \in E(G_1)$ if and only if $(a(K), a(L)) \in E(G_2)$. So for each $x \in VX_1$ there is a unique $x' \in VX_2$

such that $a(G_{1x}) = G_{2x'}$ (due to strictness) and we put $a'(x) = x'$. That is, the map $a' : VX_1 \to VX_2$ is properly defined and preserves the adjacency of the vertices. Therefore, the map a' extends to an isomorphism of trees $a' : X_1 \to X_2$.

We observe that we have $a(G_{1gx}) = G_{2a(g)a'(x)}$, so $a'(gx) = a(g)a'(x)$ for each $g \in G, x \in VX$, (2).

Now we can show that $ad(a') = a$: If $g \in G_1$, $x \in VX_1$ then from (2) we have $(ad(a')(g))(a'(x)) = a'(gx) = a(g)a'(x)$. Therefore, $ad(a')(g) = a(g)$ and so $ad(a') = a$.

Uniqueness: If $a, b : G_1 \to G_2$ are isomorphisms with $a = b$, we show that $a' = b'$. Indeed, if $x \in VX_1$, then $a(G_{1x}) = b(G_{1x})$, therefore, $G_{2a'(x)} = G_{2b'(x)}$. Thus, $a'(x) = b'(x)$ and so $a' = b'$. ∎

7. A Topological Rigidity Theorem

Let X be a tree with $G = \mathrm{Aut}(X)$, which has a countable number of edges incident at each vertex.

In this section, we extend the Rigidity Theorem (6.9) even in the case where $i_G(e) \geq 2$ for every edge e of X.

In the case where there is $e \in EX$ such that $i_G(e) = 1$, we can easily construct finite trees which are totally different but still have a trivial automorphism group.

We have $i_G(e) \geq 2$ for all $e \in VX$, then the Rigidity Theorem does not always hold. For example, we can take a tree X with $G = \mathrm{Aut}(X)$ and $i_G(e) \geq 2$ for all $e \in VX$. If $\overline{X}$ is the tree obtained from X by subdividing those edges e for which there is some $g \in G$ such that $ge = \overline{e}$, then $\mathrm{Aut}(X) = \mathrm{Aut}(\overline{X})$, but $X \ncong \overline{X}$ (as long as X has inversions).

That is, if X_1, X_2 are two trees with $G_1 = \mathrm{Aut}(X_1)$, $G_2 = \mathrm{Aut}(X_2)$ where $G_1 \approx G_2$ and $i_{G_1}(e) \geq 2$ for all $e \in EX_1$, $i_{G_2}(w) \geq 2$ for all $w \in EX_2$, then X_1 is not always isomorphic to X_2.

In this section we prove that $\overline{X}_1 \simeq \overline{X}_2$.

7.1. Terminology. Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 2$ for each edge e. We define $\overline{X}$ to be the tree obtained from X by subdividing those edges e of X for which there is $g \in G$ such that $ge = \overline{e}$ (it is obvious that $\mathrm{Aut}(\overline{X}) = G$).

We define Y to be the tree with

$$VY = \{G_x : x \in V\overline{X}\} \quad \text{and} \quad EY = \{(G_x, G_y) : x, y \text{ adjacent vertices of } \overline{X}\}.$$

Obviously (4.2) holds for the tree $\overline{X}$.

Definition 7.2: Let G be a group. Then we define

(a) $\overline{A}(G)$ to be the set of subgroups $H \leq G$ satisfying:

  (i) H is maximal among unplittable subgroups of G and

  (ii) for every family $\{H_i\}_{i\in I}, |I| \geq 2$ consisting of distinguished maximal unsplittable subgroups of G with $H \neq H_i$ for all $i \in I$ it must hold that $H \subsetneq \bigcup_{i\in I} H_i$.

(b) $\overline{E}(G)$ to be the set of elements $(K, K') \in \overline{A}(G) \times \overline{A}(G)$ satisfying:

  (i) $K \neq K'$

  (ii) the group $K \cap K'$ is maximal among the supgroups of the form $L \cap L'$, with $(L, L') \in \overline{A}(G) \times \overline{A}(G)$ and $L \neq L'$ and

  (iii) $\langle K, K' \rangle = \langle K, \bigcup_{L\in\overline{A}(G)} L : K \cap L = K \cap K' \rangle$.

Proposition 7.3 (Separation vertices): *Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 2$ for each edge e. Then, if Y is a tree with $VY = \{G_x : x \in V\overline{X}\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $\overline{X}\}$, we have $\overline{A}(G) = VY$.*

Proof. Taking into account that there are no inversions on tree $\overline{X}$ and working as in Proposition 6.7 we can show that the groups G_x, $x \in V\overline{X}$ are maximal unsplittable subgroups of G. Suppose that for some $x \in V\overline{X}$, there is a family $\{H_i\}_{i\in I}, |I| \geq 2$ consisting of distinguished maximal unsplittable subgroups of G with $G_x \neq H_i$ for all $i \in I$ such that $G_x \subseteq \bigcup_{i\in I} H_i$. Since the groups H_i are unsplittable, then from Proposition 1.4, each one of H_i fixes some vertex or end of $\overline{X}$. This is a contradiction, since the group G_x contains at least one element which moves all edges of the set $E_0(x)$ ($i_G(e) \geq 2$ for each edge e). Thus $VY \subseteq \overline{A}(G)$.

Now, if $H \in \overline{A}(G)$, then from Proposition 1.4 and Theorem 5.7 we have $H = G_x$ for some x, or else H fixes a unique end ε. From Proposition 1.4 (3) and the fact that $VY \subseteq \overline{A}(G)$, it necessarily holds that $H = G_x$ for some x. Thus $\overline{A}(G) = VY$. ∎

LEMMA 7.4: *Let X be a tree with $G = \mathrm{Aut}(X)$ which has a countable number of edges incident at each vertex and $i_G(e) \geq 2$ for each edge e. Let $x, y \in V\overline{X}$ with $x \neq y$, such that the group $G_{x,y} := G_x \cap G_y$ is maximal among groups of the form $G_{a,b}$, where $a, b \in V\overline{X}$ with $a \neq b$. Then exactly one of the following cases holds:*

(I) x, y are vertices of an axis L with a sequence of vertices $(v_i)_{i\in\mathbb{Z}}$ and edges $(e_i)_{i\in\mathbb{Z}}$ where $e_i = [v_i, v_{i+1}]$, such that $orb_{G_{v_i}}(e_i) = \{e_i, \overline{e}_{i-1}\}$ for all $i \in \mathbb{Z}$ (and so $i_G(e_i) = i_G(\overline{e}_i) = 2$).

(II) x, y are vertices of a segment T with three vertices v_1, v_2, v_3 and edges $e_1 = [v_1, v_2]$, $e_2 = [v_2, v_3]$, where $orb_{G_{v_2}}(e_2) = \{e_2, \overline{e}_1\}$, $i_G(e_1) \geq 3$ and $i_G(\overline{e}_2) \geq 3$.

(III) x, y are adjacent vertices.

Proof. Let $x = v_1, v_2, \ldots, v_n = y$ be the sequence of the vertices of the path $[x, y]$ and $e_i = [v_i, v_{i+1}]$, $i = 1, \ldots, n-1$.
Obviously we have $G_{x,y} \leq G_{x,v_i}$ for all $i = 2, \ldots, n-1$ and so $G_{x,y} = G_{x,v_i}$ for all $i = 2, \ldots, n-1$ and therefore $orb_{G_{v_i}}(e_i) = \{e_i, \overline{e}_{i-1}\}$ for all $i = 2, \ldots, n-1$, $orb_{G_{v_{i+1}}}(\overline{e}_i) = \{e_{i+1}, \overline{e}_i\}$, for all $i = 1, \ldots, n-2$. If $n \geq 4$, then necessarily Case (I) holds. If $n \leq 3$, then Case (I), Case (II) or Case (III) can hold. ∎

Remark 7.5: Keeping the above notation, we set $G_L = \langle G_{v_i} : i \in \mathbb{Z}\rangle$ the group which is generated by the vertex groups of the vertices of axis L and $G_T = \langle G_{v_1}, G_{v_2}, G_{v_3}\rangle$ the group which is generated by the vertex groups of the vertices of segment T.

Now, we work on the case of axis L. We prove that $\langle G_{v_0}, G_{v_1}\rangle = G_L$. Indeed, we have $G_{v_2} = G_{v_0}^a$ for an appropriate $a \in G_{v_1}$. Then every $g \in G_{v_2}$ is written in the form $g = a^{-1}ha$ for an appropriate $h \in G_{v_0}$. Therefore $G_{v_2} \subseteq \langle G_{v_0}, G_{v_1}\rangle$.

Now, similarly $G_{v_3} \subseteq \langle G_{v_1}, G_{v_2}\rangle \subseteq \langle G_{v_0}, G_{v_1}\rangle$. Thus inductively $G_{v_i} \subseteq \langle G_{v_0}, G_{v_1}\rangle$ for all $i \geq 0$. The proof is similar in the case where $i \leq -1$ and thus $\langle G_{v_0}, G_{v_1}\rangle = G_L$. Obviously, $G_{v_k} \nsubseteq \langle G_{v_i}, G_{v_j}\rangle$ for $i < k < j$ and so, we have$\langle G_{v_i}, G_{v_j}\rangle = G_L$ if and only if $j = i+1$, that is only groups of adjacent vertices can generate the group G_L. The same result also holds for the segment T, that is $\langle G_{v_1}, G_{v_2}\rangle = \langle G_{v_2}, G_{v_3}\rangle = G_T$ and $\langle G_{v_1}, G_{v_3}\rangle \neq G_T$.

PROPOSITION 7.6: *Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 2$ for* *each edge e. Then, if Y is a tree with $VY = \{G_x : x \in V\overline{X}\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $\overline{X}\}$, we have $\overline{E}(G) = EY$.*

Proof. Let $(G_x, G_y) \in E(Y)$. Since $i_G(e) \geq 2$ for each edge e, we have $G_x \neq G_y$. The group $G_{x,y}$ is maximal among subgroups of the form $G_{a,b}$ with $a, b \in V\overline{X}$ and $a \neq b$. Indeed, let a, $b \in V\overline{X}$ with $a \neq b$ and let $G_{x,y} \leq G_{a,b}$. If $\{x, y\} = \{a, b\}$, then we obviously have that $G_{x,y} = G_{a,b}$. If there are only three distinguished vertices among x, y, a, b, for example $x = a$, then the vertices x, y, b are vertices of a segment T or an axis L as in Lemma 7.4.

If the vertices x, y, a, b are distinguished, then these vertices are vertices of an axis L as in Lemma 7.4 (I). Using the properties of the segment T or axis L we can easily prove that $G_{x,y} = G_{a,b}$. Let $z \in V\overline{X}$ with $G_{x,y} = G_{x,z}$. Then the vertices x, y, z belong to an axis L or to a segment T as in Lemma 7.4. Let z be a vertex of L or T, different from x, y, then $G_{x,y} = G_{x,z}$ and so from (7.5) we have $\langle G_x, G_y \rangle = \langle G_x, \bigcup_{z \in V\overline{X}} G_z : G_{x,y} = G_{x,z} \rangle$. Therefore $E(Y) \subseteq \overline{E}(G)$.

Finally, $\overline{E}(G) \subseteq E(Y)$ follows from Lemma 7.4 and Remark 7.5. ∎

Now, if we work as in Teorem 6.9 for trees $\overline{X}, \overline{X}_1, \overline{X}_2$ in place of trees X, X_1, X_2 respectively, then the next Theorem follows immediately.

Topological Rigidity Theorem 7.7: I) Let X be a tree and let $G = \mathrm{Aut}(X)$. We suppose that X has a countable number of edges incident at each vertex and $i_G(e) \geq 2$ for each edge e. Then, if Y is a tree with $VY = \{G_x : x \in V\overline{X}\}$ and $EY = \{(G_x, G_y) : x, y$ adjacent vertices of $\overline{X}\}$, we have:

(a) $\overline{A}(G) = VY$

(b) $\overline{E}(G) = EY$

(c) The map $\sigma : \overline{X} \to Y$ with $\sigma(x) = G_x$ is an isomorphism of G-trees (That is, the group G determines the tree $\overline{X}$)

(d) The map $ad : G \to \mathrm{Aut}(G)$ is an isomorphism.

II) Let X_1, X_2 be trees with $G_1 = \mathrm{Aut}(X_1)$, $G_2 = \mathrm{Aut}(X_2)$ such that $i_{G_1}(e) \geq 2$ for each $e \in EX_1$ and $i_{G_2}(w) \geq 2$ for each $w \in EX_2$. Then if $a : G_1 \to G_2$ is a group isomorphism, there is a unique tree isomorphism $\sigma : \overline{X}_1 \to \overline{X}_2$ such that $a = ad(\sigma)$.

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